Unsteady Flow of an Electrically Conducting, Viscous and Compressible Fluid near an Infinite Flat Plate

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1. Introduction

The subjects of flows of an electrically conducting, viscous and compressible fluid in the presence of the external magnetic field have been investigated by several authors. Bleviss[1] discussed exactly the Couette flow problem in the presence of the transverse magnetic field in connection with the practical problems of hypersonic vehicle. In the same stand Meyer[2] discussed the flow near the stagnation point of a blunt-nosed hypersonic vehicle considering the possibility of reducing heat-transfer rate with the assumptions that the induced magnetic field can be neglected compared with the externally applied magnetic field and that the electric field is zero.

In this paper, we study the unsteady flow of an electrically conducting, viscous and compressible fluid in the externally applied transverse magnetic field near an infinite flat plate which starts to move from rest in its own direction impulsively with a uniform velocity. We assume the induced magnetic field and the electric field to be negligible. In classical hydrodynamics the same problem is investigated by Illingworth[3] applying the boundary layer approximation. In the following discussion we will apply the boundary layer approximation as Illingworth' discussion and assume the pressure to be a constant in the whole flow field. The distribution of the velocity and of the enthalpy at the commencement of motion are obtained by the successive approximation method, using the von Mises transformation and expanding in power series. The skin friction and the heat-transfer rate are calculated, where we find the effect of the magnetic field on the heat-transfer rate is more remarkable than on the skin friction.

2. Fundamental Equations

The magnetohydrodynamic equations neglecting the electric displacement current, the convection current, and excess charges are:

equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho V) = 0, \tag{2.1}$$

modified Navier-Stokes' equation

$$\rho \left(\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \operatorname{grad} \boldsymbol{V} \right) = \operatorname{div} \boldsymbol{P} + \boldsymbol{j} \times \boldsymbol{B}, \tag{2.2}$$

equation of energy

$$\rho \left(\frac{\partial H}{\partial t} + \boldsymbol{V} \cdot \operatorname{grad} H \right) = \frac{\partial p}{\partial t} + \boldsymbol{V} \cdot \operatorname{grad} p + \operatorname{div} Q + \Phi + \frac{j^2}{\sigma}, \tag{2.3}$$

Maxwell's equation

$$\operatorname{rot} \boldsymbol{E} = - \frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.4}$$

$$\operatorname{rot} \boldsymbol{H} = \boldsymbol{j}, \tag{2.5}$$

$$\operatorname{div} \boldsymbol{D} = 0, \tag{2.6}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{2.7}$$

Ohm's law

$$\boldsymbol{j} = \sigma (\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) \tag{2.8}$$

complementary relations

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \tag{2.9}$$

$$\boldsymbol{B} = \mu_m \boldsymbol{H} \tag{2.10}$$

where ρ and $\boldsymbol{V}$ are the density and the velocity of the fluid, respectively, $\boldsymbol{P}$ the mechanical stress tensor, $\boldsymbol{j}$ the electric current density, $\boldsymbol{B}$ the magnetic flux density, H the enthalpy of the fluid, p the pressure of the fluid, $\boldsymbol{Q}$ the heat flux vector, Φ the viscous dissipation energy, σ the electric conductivity, $\boldsymbol{E}$ the electric field, $\boldsymbol{H}$ the magnetic field, and $\boldsymbol{D}$ the electric displacement. ε and μ_m are the dielectric constant and the magnetic permeability, respectively. We assume the fluid to be homogeneous and isotropic. In the above equations we adopt the rationalized M. K. S. units.

Considering the unsteady motion of an infinite flat plate in an electrically conducting, viscous and compressible fluid in the presence of the externally applied transverse magnetic field, we take the coordinate the x-axis coinciding with the flat plate and the y-axis perpendicular to it. We assume here all physical variables do not depend on the variables x and z, since the flow is a plane one and the plate is infinite. The assumption that the electric field is zero since no applied or polarlization voltages exist gives $\frac{\partial B_y}{\partial t} = 0$ and $\frac{\partial B_y}{\partial y} = 0$ from (2.4) and (2.7). These means that the transverse component of the magnetic flux density is uniform in space and time. We denote it B_0. From the assumption that the induced magnetic field can be neglected compared with the externally applied magnetic field since the fluid is at best a poor electrical conductor, the pondermotive force appeared in the modified Navier-Stokes' equation and Joulean dissipation energy in the energy equation become as follows:

$$F_x = (\boldsymbol{j} \times \boldsymbol{B})_x = -\sigma B_0^2 u , \tag{2.11}$$

$$F_y = (\boldsymbol{j} \times \boldsymbol{B})_y = 0 ,$$

$$\frac{j^2}{\sigma} = \sigma B_0^2 u^2 . \tag{2.12}$$

Taking account of these results and applying the boundary layer approximation to (2.1), (2.2), and (2.3), the governing equations for this flow are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y}(\rho v) = 0, \tag{2.13}$$

$$\rho\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \sigma B_0^2 u, \tag{2.14}$$

$$\frac{\partial p}{\partial y} = 0, \tag{2.15}$$

$$\rho\left(\frac{\partial H}{\partial t} + v\frac{\partial H}{\partial y}\right) = \frac{1}{P_r}\frac{\partial}{\partial y}\left(\mu\frac{\partial H}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 + \sigma B_0^2 u^2, \tag{2.16}$$

where u and v are the x and y component of the velocity, respectively, μ the coefficient of viscosity. And moreover, P_r is Prandtl number defined by

$$P_r = \frac{\mu c_p}{\kappa}$$

where κ is the coefficient of heat conductivity, c_p the specific heat at constant pressure. In this discussion we assume Prandtl number P_r to be a constant.

Since the plate starts to move from rest in its own direction at time $t = 0$ with a uniform velocity u_0, the boundary conditions are

$$\begin{array}{lll} 0 \leq y \leq \infty ; & u = 0 , & \text{at} \quad t \leq 0 \\ y = 0 ; & u = u_0 , & \\ & H = H_w , & \text{at} \quad t > 0 \\ y \to \infty ; & u \to 0 , & \\ & H \to H_\infty , & \end{array} \tag{2.17}$$

where H_w and H_∞ are the values of the enthalpy at the plate and at a far distance from the plate, respectively. The modification to the conventional equations given by Illingworth are a $-\sigma B_0^2 u$ term in (2.14) and a $\sigma B_0^2 u^2$ term in (2.16).

3. Solutions

Solving (2.13), (2.14), (2.15) and (2.16) subjected to (2.17), we will employ the von Mises' transformation. We introduce a stream function ψ defined by

$$\frac{\partial\psi}{\partial y} = \rho\,, \qquad \frac{\partial\psi}{\partial t} = -\rho\, v\,, \tag{3.1}$$

and, therefore, (2.13) is automatically satisfied. Using t and ψ instead of t and y as independent variables, the von Mises' transformation is

$$\left(\frac{\partial}{\partial t}\right)_y = \left(\frac{\partial}{\partial t}\right)_\psi - \rho\, v \left(\frac{\partial}{\partial\psi}\right)_t\,, \tag{3.2}$$

$$\left(\frac{\partial}{\partial y}\right)_t = \rho\left(\frac{\partial}{\partial\psi}\right)_t\,. \tag{3.3}$$

Inserting them into (2.14) and (2.16), we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial\psi}\left(\mu\rho\,\frac{\partial u}{\partial\psi}\right) - \frac{\sigma B_0^2}{\rho}\,, \tag{3.4}$$

$$\frac{\partial H}{\partial t} = \frac{1}{P_r}\,\frac{\partial}{\partial\psi}\left(\mu\rho\,\frac{\partial H}{\partial\psi}\right) + \mu\rho\left(\frac{\partial u}{\partial\psi}\right)^2 + \frac{\sigma B_0^2}{\rho}\,u^2\,. \tag{3.5}$$

Since a $-\frac{\sigma B_0^2}{\rho}$ term and a $\frac{\sigma B_0^2}{\rho}u^2$ term are included in above equations, there occurs the intricateness in the mathematical treatment. In order to enable us to analyze, we assume: 1) the pressure is a constant in the whole flow field, 2) the coefficient of viscosity is proportional to the temperature of the fluid, 3) the specific heat is a constant, 4) the electric conductivity is a constant and independent on the temperature. In the above assumptions, 4) can be interpreted from the fact that the electric conductivity of the fluid is intrinsically a function of temperature, but the dependence on the temperature becomes smaller by taking account of seeding effect[4).

From the assumptions 1), 3), and 4), $\frac{\sigma B_0^2}{\rho}$ in (3.4) and (3.5) can be described as

$$\frac{\sigma B_0^2}{\rho} = \frac{\sigma B_0^2}{\rho_w}\,\frac{H}{H_w}\,. \tag{3.6}$$

And hence, we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial\psi}\left(\mu\rho\,\frac{\partial u}{\partial\psi}\right) - \frac{\sigma B_0^2}{\rho_w}\,\frac{H}{H_w}\,, \tag{3.7}$$

$$\frac{\partial H}{\partial t} = \frac{1}{P_r}\,\frac{\partial}{\partial\psi}\left(\mu\rho\,\frac{\partial H}{\partial\psi}\right) + \mu\rho\left(\frac{\partial u}{\partial\psi}\right)^2 + \frac{\sigma B_0^2}{\rho_w}\,u^2\,\frac{H}{H_w}\,. \tag{3.8}$$

Here we will seek the solutions by the successive approximation method expanding in power series. Putting $\mu\rho = b$ from the assumptions 1) and 2), and introducing a new variable $\zeta = \frac{\psi}{2\sqrt{bt}}$, we expand u and H in the following power series

$$u = u_0 \left\{ F_0(\zeta) + mt\, F_1(\zeta) + (mt)^2 F_2(\zeta) + \cdots\cdots \right\}, \tag{3.9}$$

$$H = H_w \left\{ G_0(\zeta) + mt\, G_1(\zeta) + (mt)^2 G_2(\zeta) + \cdots\cdots \right\}, \tag{3.10}$$

where m is a magnetic parameter and is defined by

$$m = \frac{\sigma B_0^2}{\rho_w} .$$

Inserting them into (3.7) and (3.8), we have the following system of differential equations

$$\frac{d^2F_0}{d\zeta^2} + 2\,\zeta\,\frac{dF_0}{d\zeta} = 0, \tag{3.11}$$

$$\frac{d^2F_1}{d\zeta^2} + 2\,\zeta\,\frac{dF_1}{d\zeta} - 4\,F_1 - 4\,F_0\,G_0 = 0, \tag{3.12}$$

....

$$\frac{d^2G_0}{d\zeta^2} + 2\,P_r\zeta\,\frac{dG_0}{d\zeta} + P_r\left(\frac{dF_0}{d\zeta}\right)^2 = 0, \tag{3.13}$$

$$\frac{d^2G_1}{d\zeta^2} + 2\,P_r\zeta\,\frac{dG_1}{d\zeta} - 4\,P_r\,G_1 + 2\,P_r\,\frac{dF_0}{d\zeta}\,\frac{dF_1}{d\zeta} + 4\,P_r\,F_0^2\,G_0 = 0, \tag{3.14}$$

....

The boundary conditions for them become

$$F_0(o) = 1, \quad F_0(\infty) = 0, \tag{3.15}$$

$$F_1(o) = F_1(\infty) = \cdots\cdots = 0, \tag{3.16}$$

$$G_0(o) = G_w,\ G_0(\infty) = G_\infty, \tag{3.17}$$

$$G_1(o) = G_1(\infty) = \cdots\cdots = 0. \tag{3.18}$$

The solution of (3.11) subjected to the boundary condition (3.15) can be given by

$$F_0(\zeta) = 1 - \operatorname{erf}\zeta \tag{3.19}$$

where erf ζ is the error function defined by

$$\operatorname{erf}\zeta = \frac{2}{\sqrt{\pi}}\int_0^{\zeta} e^{-\xi^2}\,d\xi .$$

Inserting this solution into (3.13), we have the solution of (3.13) as

$$G_0(\zeta) = A + B\operatorname{erf}(\zeta\sqrt{P_r}) - \frac{2}{\sqrt{\pi(2-P_r)P_r}}\int_0^{\zeta\sqrt{P_r}} e^{-\xi^2}\operatorname{erf}\left(\xi\sqrt{\frac{2-P_r}{P_r}}\right)d\xi \tag{3.20}$$

where A and B are arbitrary constants, and can be determined by (3.17). These solutions as the first approximation in our successive procedure agree with the results given by Illingworth in the case of $\mu\rho$ = *const.* Inserting (3.19) and (3.20) into (3.12) in order to find the solution of u as the second approximation, we get the following differential equation

$$\frac{d^2F_1}{d\zeta^2}+2\zeta\frac{dF_1}{d\zeta}-4F_1=4(1-\mathrm{erf}\zeta)\{A+B\mathrm{erf}(\zeta\sqrt{P_r})$$

$$-\frac{2}{\sqrt{\pi(2-P_r)P_r}}\int_0^{\zeta\sqrt{P_r}}e^{-\xi^2}\,\mathrm{erf}\left(\xi\sqrt{\frac{2-P_r}{P_r}}\right)d\xi \quad (3.21)$$

The solution of (3.21) can be described as

$$F_1(\zeta)=\alpha(2\zeta^2+1)+\beta\left\{(2\zeta^2+1)\,\mathrm{erf}\,\zeta+\frac{2}{\sqrt{\pi}}\zeta e^{-\zeta^2}\right\}$$

$$-(2\zeta^2+1)\int_0^{\zeta}\frac{\sqrt{\pi}}{4}e^{\xi^2}\left\{(2\xi^2+1)\,\mathrm{erf}\,\xi+\frac{2}{\sqrt{\pi}}\xi e^{-\xi^2}\right\}f(\xi)d\xi \quad (3.22)$$

$$+\left\{(2\zeta^2+1)\,\mathrm{erf}\,\zeta+\frac{2}{\sqrt{\pi}}\zeta e^{-\zeta^2}\right\}\int_0^{\zeta}\frac{\sqrt{\pi}}{4}e^{\xi^2}(2\xi^2+1)f(\xi)d\xi,$$

where

$$f(\xi)=4(1-\mathrm{erf}\xi)\left\{A+B\,\mathrm{erf}(\xi\sqrt{P_r})-\frac{2}{\sqrt{\pi(2-P_r)P_r}}\int_0^{\xi\sqrt{P_r}}e^{-\eta^2}\,\mathrm{erf}\left(\eta\sqrt{\frac{2-P_r}{P_r}}\right)d\eta,\right.$$

and an arbitrary constant α and β are determined by (3.16).

4. Solutions at P_r = 1

We find the successive solutions obtained in the previous section are an integral form. In order to find the distribution of velocity and of enthalpy, we must integrate them numerically. However, when Prandtl number P_r is unity an analytical procedure is possible.

Putting $P_r = 1$, the solutions as the first approximation are expressed as

$$F_0(\zeta)=1-\mathrm{erf}\,\zeta, \quad (4.1)$$

$$G_0(\zeta)=A+B\,\mathrm{erf}\,\zeta-\frac{1}{2}\mathrm{erf}^2\zeta, \quad (4.2)$$

where arbitrary constants A and B are determined by (3.17) as

$$A=G_w,$$

$$B=\frac{1}{2}-G_w+G_\infty. \quad (4.3)$$

Obtaining the solution of $F_1(\zeta)$ as the second approximation for u, we insert (4.1) and (4.2) into (3.12). We have

$$\frac{d^2F_1}{d\zeta^2} + 2\zeta\frac{dF_1}{d\zeta} - 4F_1$$

$$= 2\ \mathrm{erf}^3\zeta - 4\left(B + \frac{1}{2}\right)\mathrm{erf}^2\zeta + 4(B-A)\ \mathrm{erf}\ \zeta + 4A. \tag{4.4}$$

Solving (4.4), we assume the solution to be

$$F_1(\zeta) = \alpha(2\zeta^2+1) + \beta\left\{(2\zeta^2+1)\ \mathrm{erf}\ \zeta + \frac{2}{\sqrt{\pi}}\zeta e^{-\zeta^2}\right\}$$

$$+ X_1(\zeta)\ \mathrm{erf}^3\zeta + Y_1(\zeta)\ \mathrm{erf}^2\zeta + Z_1(\zeta)\ \mathrm{erf}\ \zeta + T(\zeta), \tag{4.5}$$

where $X_1(\zeta)$, $Y_1(\zeta)$, $Z_1(\zeta)$ and $T_1(\zeta)$ are assumed to be a function of ζ, in which the powers of the error function are not involved. Here we introduce a differential operator defined by[6)]

$$P = \frac{d^2}{d\zeta^2} + 2\zeta\frac{d}{d\zeta} - 4 \tag{4.6}$$

and operate it to (4.5). The result is

$$\begin{aligned} P\left\{F_1(\zeta)\right\} &= P(X_1)\ \mathrm{erf}^3\zeta + \left\{P(Y_1) + \frac{12}{\sqrt{\pi}}X_1'e^{-\zeta^2}\right\}\mathrm{erf}^2\zeta \\ &+ \left\{P(Z_1) + \frac{24}{\pi}X_1e^{-2\zeta^2} + \frac{8}{\sqrt{\pi}}Y_1'e^{-\zeta^2}\right\}\mathrm{erf}\ \zeta \\ &+ P(T_1) + \frac{8}{\pi}Y_1e^{-2\zeta^2} + \frac{4}{\sqrt{\pi}}Z_1'e^{-\zeta^2}, \end{aligned} \tag{4.7}$$

where the prime denotes the differentiation to ζ. Equating (4.4) with (4.7), we have the following differential equations:

$$P(X_1) = 2, \tag{4.8}$$

$$P(Y_1) + \frac{12}{\sqrt{\pi}}X_1'e^{-\zeta^2} = -4\left(B + \frac{1}{2}\right), \tag{4.9}$$

$$P(Z_1) + \frac{24}{\pi}X_1e^{-2\zeta^2} + \frac{8}{\sqrt{\pi}}Y_1'e^{-\zeta^2} = 4(B-A), \tag{4.10}$$

$$P(T_1) + \frac{8}{\pi}Y_1e^{-2\zeta^2} + \frac{4}{\sqrt{\pi}}Z_1'e^{-\zeta^2} = 4A. \tag{4.11}$$

The solution of (4.8) is

$$X_1(\zeta) = c_1(2\zeta^2+1) - \frac{1}{2}, \tag{4.12}$$

where c_1 is an arbitrary constant. Inserting (4.12) into (4.9), (4.9) becomes

$$\frac{d^2Y_1}{d\zeta^2} + 2\zeta\frac{dY_1}{d\zeta} - 4Y_1 = -\frac{48}{\sqrt{\pi}}c_1\zeta e^{-\zeta^2} - 4\left(B + \frac{1}{2}\right). \tag{4.13}$$

The solution of (4.13) can be described as

$$Y_1(\zeta) = c_2(2\zeta^2+1) + \frac{6}{\sqrt{\pi}}c_1\zeta e^{-\zeta^2} + B + \frac{1}{2}. \tag{4.14}$$

Again inserting (4.12) and (4.14) into (4.10), we have

$$\frac{d^2Z_1}{d\zeta^2} + 2\zeta\frac{dZ_1}{d\zeta} - 4Z_1 = -\frac{32}{\sqrt{\pi}}c_2\zeta e^{-\zeta^2} + \frac{48}{\pi}c_1\zeta^2 e^{-2\zeta^2}$$

$$-\frac{24}{\pi}\left(3c_1 - \frac{1}{2}\right)e^{-2\zeta^2} + 4(B-A). \tag{4.15}$$

Taking account of the assumption that $Z_1(\zeta)$ does not involve the powers of the error function, we must put an arbitrary constant c_1 as $c_1 = \frac{1}{2}$. Then the solution of (4.15) becomes

$$Z_1(\zeta) = c_3(2\zeta^2+1) + \frac{4}{\sqrt{\pi}}c_2\zeta e^{-\zeta^2} + \frac{3}{\pi}e^{-2\zeta^2}$$
$$-(B-A). \tag{4.16}$$

Repeating this procedure which inserts the solutions already obtained into the next differential equation, we have the following differential equation for (4.11):

$$\frac{d^2T_1}{d\zeta^2} + 2\zeta\frac{dT_1}{d\zeta} - 4T_1 = \frac{16}{\pi}c_2\zeta^2 e^{-2\zeta^2} - \frac{8}{\pi}\left(3c_2 + B + \frac{1}{2}\right)e^{-2\zeta^2}$$

$$+\frac{24}{\sqrt{\pi^3}}\zeta e^{-3\zeta^2} - \frac{16}{\sqrt{\pi}}c_3\zeta e^{-\zeta^2} + 4A. \tag{4.17}$$

From the assumption that $T_1(\zeta)$ does not involve the powers of the error function, we must put $c_2 = -\left(B + \frac{1}{2}\right)$. Therefore, the solution of (4.17) is

$$T_1(\zeta) = -\frac{2}{\pi}\left(B + \frac{1}{2}\right)e^{-2\zeta^2} + \frac{2}{\sqrt{\pi}}c_3\zeta e^{-\zeta^2}$$

$$-\frac{3\sqrt{3}}{2\pi}(2\zeta^2+1)\,\mathrm{erf}\,\zeta\sqrt{3} - \frac{3}{\sqrt{\pi^3}}\zeta e^{-3\zeta^2} - A \tag{4.18}$$

where the homogeneous solution $2\zeta^2 + 1$ does not appear because it already appeared in (4.5). The arbitrary constant α , β and c_3 can be determined by the boundary condition (3.16). The constants become

$$\alpha = A + \frac{2}{\pi} B + \frac{1}{\pi} ,$$

$$\beta = \frac{3\sqrt{3}}{2\pi} + B - A - \frac{2}{\pi} B - \frac{1}{\pi} ,$$

$$c_3 = 0 .$$

After all, the solution as the second approximation for u can be described as

$$F_1(\zeta) = \alpha(2\zeta^2+1) + \beta \left\{(2\zeta^2+1)\ \mathrm{erf}\ \zeta + \frac{2}{\sqrt{\pi}} \zeta\, e^{-\zeta^2}\right\}$$

$$+ X_1(\zeta)\ \mathrm{erf}^3\ \zeta + Y_1(\zeta)\ \mathrm{erf}^2\ \zeta + Z_1(\zeta)\ \mathrm{erf}\ \zeta + T(\zeta) , \qquad (4.19)$$

where

$$X_1(\zeta) = \zeta^2 ,$$

$$Y_1(\zeta) = -2\left(B + \frac{1}{2}\right)\zeta^2 + \frac{3}{\pi}\zeta\, e^{-\zeta^2} ,$$

$$Z_1(\zeta) = -\frac{4}{\sqrt{\pi}}\left(B + \frac{1}{2}\right)\zeta\, e^{-\zeta^2} + \frac{3}{\pi} e^{-2\zeta^2} + A - B ,$$

$$T_1(\zeta) = -\frac{2}{\pi}\left(B + \frac{1}{2}\right) e^{-2\zeta^2} - \frac{3\sqrt{3}}{2\pi}(2\zeta^2+1)\ \mathrm{erf}\ \zeta\sqrt{3}$$

$$- \frac{3}{\sqrt{\pi^3}} \zeta\, e^{-3\zeta^2} - A,$$

and the arbitrary constants are

$$\alpha = A + \frac{2}{\pi} B + \frac{1}{\pi} ,$$

$$\beta = \frac{3\sqrt{3}}{2\pi} + B - A - \frac{2}{\pi} B - \frac{1}{\pi} .$$

The second approximation for the enthalpy is carried out as follows. We insert the first approximate solution (4.1) and (4.2) and the second approximate solution (4.9) into (3.14). The differential equation (3.14) becomes

$$\frac{d^2 G_1}{d\zeta^2} + 2\zeta \frac{dG_1}{d\zeta} - 4G_1 = 2\ \mathrm{erf}^4\zeta + \left\{\frac{8}{\sqrt{\pi}}\zeta e^{-\zeta^2} - 4(B+1)\right\}\mathrm{erf}^3\ \zeta$$

$$+ \left[\frac{4}{\sqrt{\pi}}\left\{\frac{3}{\sqrt{\pi}} e^{-\zeta^2} - 4\left(B + \frac{1}{2}\right)\zeta\right\} e^{-\zeta^2} - 4\left(A - 2B - \frac{1}{2}\right)\right]\mathrm{erf}^2\ \zeta$$

$$+\left[\frac{4}{\sqrt{\pi}}\left\{4\beta\zeta-\frac{4}{\sqrt{\pi}}\left(B+\frac{1}{2}\right)e^{-\zeta^2}\right\}e^{-\zeta^2}-4(B-2A)\right]\operatorname{erf}\zeta$$

$$+\frac{4}{\sqrt{\pi}}\left\{4\alpha\zeta+\frac{4}{\sqrt{\pi}}\beta e^{-\zeta^2}-\frac{2}{\sqrt{\pi}}(B-A)e^{-\zeta^2}\right. \tag{4.20}$$

$$\left.-\frac{6}{\sqrt{\pi^3}}e^{-3\zeta^2}-\frac{6\sqrt{3}}{\pi}\zeta\operatorname{erf}\zeta\sqrt{3}\right\}e^{-\zeta^2}-4A.$$

The procedure obtaining the solution subjected to the boundary condition (3.18) is similar to the one in obtaining the second approximate solution of u. We assume the solution of $G_1(\zeta)$ to be

$$G_1(\zeta)=\gamma(2\zeta^2+1)+\delta\left\{(2\zeta^2+1)\operatorname{erf}\zeta+\frac{2}{\sqrt{\pi}}\zeta e^{-\zeta^2}\right\}$$

$$+W_2(\zeta)\operatorname{erf}^4\zeta+X_2(\zeta)\operatorname{erf}^3\zeta+Y_2(\zeta)\operatorname{erf}^2\zeta$$

$$+Z_2(\zeta)\operatorname{erf}\zeta+T_2(\zeta) \tag{4.21}$$

where again $W_2(\zeta)$, $X_2(\zeta)$, $Y_2(\zeta)$, $Z_2(\zeta)$ and $T_2(\zeta)$ are assumed to be a function of ζ, in which the powers of the error function are not involved. Operating the differential operator defined by (4.6) to (4.21), (4.21) is split into the following differential equations:

$$P(W_2)=2, \tag{4.22}$$

$$P(X_2)+\frac{16}{\sqrt{\pi}}W_2'e^{-\zeta^2}=\frac{8}{\sqrt{\pi}}\zeta e^{-\zeta^2}-4(B+1), \tag{4.23}$$

$$P(Y_2)+\frac{12}{\sqrt{\pi}}X_2'e^{-\zeta^2}+\frac{48}{\pi}W_2e^{-2\zeta^2}$$

$$=\frac{4}{\sqrt{\pi}}\left\{\frac{3}{\sqrt{\pi}}e^{-\zeta^2}-4\left(B+\frac{1}{2}\right)\zeta\right\}e^{-\zeta^2}-4\left(A-2B-\frac{1}{2}\right), \tag{4.24}$$

$$P(Z_2)+\frac{8}{\sqrt{\pi}}Y_2'e^{-\zeta^2}+\frac{24}{\pi}X_2e^{-\zeta^2} \tag{4.25}$$

$$=\frac{4}{\sqrt{\pi}}\left\{4\beta\zeta-\frac{4}{\sqrt{\pi}}\left(B+\frac{1}{2}\right)e^{-\zeta^2}\right\}e^{-\zeta^2}-4(B-2A),$$

$$P(T_2)+\frac{4}{\sqrt{\pi}}Z_2'e^{-\zeta^2}+\frac{8}{\pi}Y_2e^{-2\zeta^2}$$

$$=\frac{4}{\sqrt{\pi}}\left\{4\alpha\zeta+\frac{4}{\sqrt{\pi}}\beta e^{-\zeta^2}-\frac{2}{\sqrt{\pi}}(B-A)e^{-\zeta^2}-\frac{6}{\sqrt{\pi^3}}e^{-3\zeta^2}\right. \tag{4.26}$$

$$\left.-\frac{6\sqrt{3}}{\pi}\zeta\operatorname{erf}\zeta\sqrt{3}\right\}e^{-\zeta^2}-4A.$$

We solve them successively determining the arbitrary constants to be subjected to the assumption that the function of ζ do not involve the powers of the error function, and to the boundary condition (3.18). The solution as the second approximation for H finally can be given by

$$
\begin{aligned}
G_1(\zeta) = {} & \gamma\,(2\zeta^2+1) + \delta\left\{(2\zeta^2+1)\operatorname{erf}\zeta + \frac{2}{\sqrt{\pi}}\,\zeta\, e^{-\zeta^2}\right\} \\
& + W_2(\zeta)\operatorname{erf}^4\zeta + X_2(\zeta)\operatorname{erf}^3\zeta + Y_2(\zeta)\operatorname{erf}^2\zeta \\
& + Z_2(\zeta)\operatorname{erf}\zeta + T_2(\zeta),
\end{aligned}
\tag{4.27}
$$

where

$$W_2(\zeta) = \zeta^2,$$

$$X_2(\zeta) = -2\,(B+1)\,\zeta^2 + \frac{3}{\sqrt{\pi}}\,\zeta\, e^{-\zeta^2},$$

$$
\begin{aligned}
Y_2(\zeta) = {} & -\frac{2}{\sqrt{\pi}}\left(2B+\frac{5}{2}\right)\zeta\, e^{-\zeta^2} + \frac{3}{\pi}\, e^{-2\zeta^2} + \left(\beta + B + \frac{1}{2}\right)(2\zeta^2+1) \\
& + A - 2B - \frac{1}{2},
\end{aligned}
$$

$$
\begin{aligned}
Z_2(\zeta) = {} & \frac{2}{\sqrt{\pi}}\,(\beta+2B+1)\,\zeta\, e^{-\zeta^2} - \frac{1}{\pi}\,(2B+4)\, e^{-2\zeta^2} \\
& - \frac{3\sqrt{3}}{2\pi}\,(2\zeta^2+1)\operatorname{erf}\zeta\sqrt{3} - \frac{3}{\sqrt{\pi^3}}\,\zeta\, e^{-3\zeta^2} + B - 2A,
\end{aligned}
$$

$$
\begin{aligned}
T_2(\zeta) = {} & -\frac{2}{\sqrt{\pi}}\,\alpha\,\zeta\, e^{-\zeta^2} + \frac{1}{\pi}\,(2B+1)\, e^{-2\zeta^2} \\
& + \frac{3\sqrt{3}}{2\pi}\,(2\zeta^2+1)\operatorname{erf}\zeta\sqrt{3} + \frac{3}{\sqrt{\pi^3}}\,\zeta\, e^{-3\zeta^2} + A,
\end{aligned}
$$

and arbitrary constants γ and δ are

$$\gamma = -\left(A + \frac{2}{\pi}B + \frac{1}{\pi}\right),$$

$$\delta = 2\left(A + \frac{2}{\pi}B + \frac{1}{\pi}\right) - \frac{3\sqrt{3}}{2\pi} - B.$$

5. On the Skin Friction and the Heat-transfer Rate for the case of $P_r = 1$

In the previous section we find the distribution of velocity and of enthalpy by successive approximation method. From these results the skin friction and the heat-transfer rate can be calculated for the case of $P_r = 1$.

The skin friction, calculated from the second approximation, is

$$\begin{aligned}\tau_w &= -\left(\mu \frac{\partial u}{\partial y}\right)_{y=0} \\ &= -\frac{u_0}{2}\sqrt{\frac{b}{t}}\left(\frac{dF}{d\zeta}\right)_{\zeta=0} \\ &= \frac{u_0}{2}\sqrt{\frac{b}{t}}\left[\frac{2}{\sqrt{\pi}} - mt\left\{\frac{4}{\sqrt{\pi}}\beta + \frac{2}{\sqrt{\pi}}\left(\frac{3}{\pi} + A - B\right) - \frac{12}{\sqrt{\pi^3}}\right\}\right].\end{aligned} \tag{5.1}$$

For the case $G_w = \frac{1}{3}$ and $G_\infty = \frac{1}{12}$, the dimensionless skin friction is

$$\tau_w{}^* = \tau_w \Big/ \frac{u_0}{2}\sqrt{\frac{b}{t}} = \frac{2}{\sqrt{\pi}} + 0.383mt. \tag{5.2}$$

We find the effect of the magnetic field increases the skin friction. On the other hand the heat-transfer rate, calculated from the second approximation, is given by

$$\begin{aligned}q &= \frac{\kappa}{c_p}\left(\frac{\partial H}{\partial y}\right)_{y=0} \\ &= \frac{u_0^2}{2}\sqrt{\frac{b}{t}}\left(\frac{dG}{d\zeta}\right)_{\zeta=0} \\ &= \frac{u_0^2}{2}\sqrt{\frac{b}{t}}\left[\!\left[\frac{2}{\sqrt{\pi}}B + mt\left[\frac{4}{\sqrt{\pi}}\delta - \frac{2}{\sqrt{\pi}}\alpha + \frac{12}{\sqrt{\pi^3}}\right.\right.\right. \\ &\qquad \left.\left.\left. + \frac{2}{\sqrt{\pi}}\left\{-\frac{1}{\pi}(2B+4)+B-2A\right\}\right]\right]\!\right].\end{aligned} \tag{5.3}$$

For the case $G_w = \frac{1}{3}$ and $G_\infty = \frac{1}{12}$, the dimensionless heat-transfer rate is

$$q^* = q\Big/\frac{u_0^2}{2}\sqrt{\frac{b}{t}} = \frac{1}{2\sqrt{\pi}} + 0.383mt. \tag{5.4}$$

The heat-transfer rate also increases by the effect of the magnetic field.

Investigating whether the effect of the magnetic field contributes more remarkable to the skin friction, or to the heat-transfer rate, we calculate the ratio of the magnetohydrodynamical case to the ordinary hydrodynamical case both on the skin friction and on the heat-transfer rate. The results are tabulated in Table, which shows that the contribution to the latter is more dominant to the former. These results are also true for arbitrary values of A and B , because we happen to find the relation

$$\left| \left(\frac{dF_1}{d\zeta} \right)_{\zeta=0} \right| = \left| \left(\frac{dG_1}{d\zeta} \right)_{\zeta=0} \right| . \tag{5.5}$$

This conclusion can be interpreted from the fact that the viscous dissipation due to the skin frictional drag increases by the effect of the magnetic field.

Table. The effect of the magnetic field on the skin friction and the heat-transfer rate (compared with the ordinary hydrodynamical results) :

mt	0.0	0.2	0.4	0.6	0.8	1.0
τ^*_w/τ^*_{wo}	1.000	1.194	1.387	1.581	1.775	1.969
$q^*/q_0{}^*$	1.000	1.775	2.550	3.325	4.100	4.875

($\tau_w{}^*{}_o$ and $q_0{}^*$ denote the skin friction and the heat-transfer rate in the ordinary hydrodynamics, respectively:)

In the previous investigations by Bleviss[1] and Meyer[2], it has been concluded that the contribution of the magnetic field to the total skin frictional drag involving the magnetic skin frictional drag is more remarkable than the one to the heat-transfer rate. This conclusion in contrast to ours is caused by the electromagnetic property of the body. Both investigations are concerned with the case when the plate is an electrically perfect conductor. In the investigation of Rayleigh's problem, Rossow[6] has treated by distinguishing two cases neglecting the electric field: 1) the magnetic lines of force are fixed relative to the plate, and 2) fixed relative to the fluid. It is well known that these two cases correspond to the plate when it is an electrically perfect insulator, and an electrically perfect conductor, respectively. The discussion in this paper can be considered to be concerned with the former case.

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無限平板附近の電導性圧縮性粘性流体の非定常流

片　桐　理　和

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磁場が流れに垂直に与えられているもとで一様な速度で電気的伝導性を有する圧縮性粘性流体の中を運動する無限平板の近くの流れに関して調べる。流体との相互作用により誘起される磁場を無視するという仮定のもとで，平板附近の流れに境界層近似を行い級数展開により解析的に議論を進め流体の速度分布及びエンタルピーの分布が得られる。これより平板の運動による摩擦抵抗及び，流体から平板への熱伝達率を求め，磁場は前者よりも後者により多く影響を及ぼしていることが知られる。